Homework 9

Due: Wednesday, April 17, 2024

All homeworks are due at 11:59 PM on Gradescope.

Please do not include any identifying information about yourself in the handin, including your Banner ID.

Be sure to fully explain your reasoning and show all work for full credit.

Problem 1

For each of the statements below, determine if it is *always* true, *sometimes* true, or *never* true. Justify your answers. To justify an "always" or "never" answer, write a proof; to justify a "sometimes" answer, give one witness that makes the statement true and one that makes the statement false, explaining these judgments.

For example, the statement

> Let $\mathbb{P}$ be a probability function on a sample space Ω. Then $\mathbb{P}[\omega] = \mathbb{P}[\omega']$ for every pair of outcomes $\omega, \omega' \in \Omega$.

is *sometimes* true. It is true for our classic "coin flip" probability space, where $\Omega = \{H, T\}$ with $\mathbb{P}[H] = \mathbb{P}[T] = \frac{1}{2}$. It is false for a "weighted die" probability space, where $\Omega = \{1, 2, 3, 4, 5, 6\}$ with $\mathbb{P}[1] = \mathbb{P}[2] = \mathbb{P}[3] = \mathbb{P}[4] = \mathbb{P}[5] = .1$ and $\mathbb{P}[6] = .5$.

a. Let $\mathbb{P}$ be a probability function on a sample space Ω. The only event whose probability is 0 is the empty event $\emptyset \subseteq \Omega$.

b. Let $\mathbb{P}$ be a probability function on a sample space Ω and let A and B be events with $\mathbb{P}[B] > 0$. Then $\mathbb{P}[A \mid B] \geq \mathbb{P}[A]$.

c. Let $\mathbb{P}$ be a probability function on a sample space Ω and let A and B be events such that $\mathbb{P}[B] > 0$ and no outcome is in both A and B. Then $\mathbb{P}[A \mid B] = 0$.

d. Let $\mathbb{P}$ be a probability function on a sample space Ω and let A and B be independent events with $\mathbb{P}[B] > 0$. Then $\mathbb{P}[B] = \mathbb{P}[A]$.

e. Let $\mathbb{P}$ be a probability function on a finite sample space Ω and let R be a random variable on this probability space, taking values in $\mathbb{R}$. Then $\mathbb{E}[R]$ is infinite.

Solution:

a. Sometimes. True for the "fair coin flip" probability space because the only possible events are $\varnothing$, $\{H\}$, $\{T\}$, and $\{H,T\}$, which have probabilities 0, 0.5, 0.5, and 1, respectively. False for the "maximally unfair coin flip" probability space for which $\Omega = \{H,T\}$, $\mathbb{P}[H] = 1$, and $\mathbb{P}[T] = 0$, since the event $\{T\}$ has probability 0 but is nonempty. (Note that this is a valid probability space since all outcomes' probabilities are nonnegative and sum to 1.)

b. Sometimes. True in the "fair coin flip" case where $A = \{H\}$ and $B = \{H,T\} = \Omega$ (note that $\mathbb{P}[B[] = 1 > 0$), since

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap \Omega]}{\mathbb{P}[\Omega]} = \frac{\mathbb{P}[A]}{1} \geq \mathbb{P}[A].$$

False in the "fair coin flip" case where $A = \{H\}$ and $B = \{T\}$ (note $\mathbb{P}[B] = 0.5 > 0$), since

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} = \frac{\mathbb{P}[\{H\} \cap \{T\}]}{0.5} = \frac{\mathbb{P}[\varnothing]}{0.5} = 0 < \mathbb{P}[A] = 0.5.$$

c. Always. If A and B are disjoint, then $A \cap B = \varnothing$, and we know $\mathbb{P}[\varnothing] = 0$. But then we have

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} = \frac{\mathbb{P}[\varnothing]}{\mathbb{P}[B]} = \frac{0}{\mathbb{P}[B]} = 0,$$

noting that this is well defined because $\mathbb{P}[B] > 0$.

d. Sometimes.

True in the following case: let $\Omega = \{HH, HT, TH, TT\}$ and let $\mathbb{P} : \omega \to [0,1]$ be defined by $\mathbb{P}(\omega) = 0.25$ for all $\omega \in \Omega$. (This defines the "two fair coin flip" probability space.) Let $A = \{HH, HT\}$ be the event of getting heads on the first throw, and let $B = \{TH, HH\}$ be the event of getting heads on the second. Note that $\mathbb{P}[A] = \mathbb{P}[B] = 0.5$. Moreover, these are independent because

$$\mathbb{P}[A \cap B] = \mathbb{P}[\{HH\}] = 0.25 = 0.5 \cdot 0.5 = \mathbb{P}[A]\mathbb{P}[B].$$

False in the following case: let Ω, $\mathbb{P}$, and A be as above, but now take $B = \Omega$. Observe that $\mathbb{P}[A] = 0.5 \neq 1 = \mathbb{P}[B]$. But these events are independent because

$$\mathbb{P}[A \cap B] = \mathbb{P}[A] = \mathbb{P}[A] \cdot 1 = \mathbb{P}[A]\mathbb{P}[B].$$

e. Never. $\mathbb{E}[R] = \sum_{\omega \in \Omega} R(\omega)\mathbb{P}[\omega]$, a finite sum of finite numbers, so it must be finite.

Problem 2

The Weather Channel's Thursday night weather report predicts a 60% chance that it will rain tomorrow (Friday).

In general, if it rains on a given day there is an 80% chance that it will rain the next day, but if it does not rain on a given day there is only a 10% chance that it will rain the next day.

a. To answer the following questions, you will need to model the situation as a *probability space*. Read the questions below, and come up with the model you will use. What is the *sample space*? What are the *events* we will consider in parts b-d, expressed as sets of outcomes?
b. What is the probability that it will rain on Saturday?
c. What is the probability that it will rain on Sunday?
d. What is the probability that it will rain at least once this coming weekend?
e. Given that it will rain on Saturday, what is the probability that it rains tomorrow?

Solution:

Let F be the proposition that it rains on Friday, Sa be the proposition that it rains on Saturday, and S be the proposition that it rains on Sunday.

a. We can consider an outcome to be a length 3 sequence of "booleans" representing whether it rains or not on Friday, Saturday, and Sunday respectively. So our sample space is $\{0,1\}^3$, or anything equivalent; an example outcome would be $(0,0,1)$, the outcome that it rains only on Sunday.

   The event that it rains on Saturday is $\{(a,b,c) \mid a,b,c \in \{0,1\}, b = 1\}$. The event that it rains on Sunday is $\{(a,b,c) \mid a,b,c \in \{0,1\}, c = 1\}$. The event that it rains at least one day is $\{(a,b,c) \mid a,b,c \in \{0,1\}, a = 1 \vee b = 1 \vee c = 1\}$.
b. We have

$$\begin{aligned} P(Sa) &= P(F) \cdot P(Sa \mid F) + P(\neg F) \cdot P(Sa \mid \neg F) \\ &= 0.6 \cdot 0.8 + 0.4 \cdot 0.1 \\ &= \boxed{0.52} \end{aligned}$$

c. We have

$$P(S) = P(Sa) \cdot P(S \mid Sa)$$

$$\begin{aligned}&+ P(\neg Sa) \cdot P(S \mid \neg Sa)\\ &= 0.52 \cdot 0.8 + 0.48 \cdot 0.1\\ &= \boxed{0.464}\end{aligned}$$

d. We have

$$\begin{aligned}P(\text{rain at least once weekend}) &= 1 - P(\text{not rain weekend})\\ &= 1 - (P(F) \cdot P(\neg Sa \mid F) \cdot P(\neg S \mid \neg Sa)\\ &\quad + P(\neg F) \cdot P(\neg Sa \mid \neg F) \cdot P(\neg S \mid \neg Sa))\\ &= 1 - (0.6 \cdot 0.2 \cdot 0.9 + 0.4 \cdot 0.9 \cdot 0.9)\\ &= 1 - (0.108 + 0.324) = 1 - 0.432 = \boxed{0.568}\end{aligned}$$

Note that $P(\text{not rain weekend}) \neq P(\neg S) \cdot P(\neg Sa)$, because S and Sa are not independent!

e. We know $\mathbb{P}(Sa \mid F) = 0.8$ by the problem statement, $\mathbb{P}(F) = 0.6$ by the weather report, and $\mathbb{P}(Sa) = 0.52)$ by part (b). Using Bayes' Rule, we have:

$$\begin{aligned}\mathbb{P}(F \mid Sa) &= \frac{\mathbb{P}(Sa \mid F)\mathbb{P}(F)}{\mathbb{P}(Sa)}\\ &= \frac{0.8 \cdot 0.6}{0.52}\\ &\approx \boxed{0.92}\end{aligned}$$

Problem 3

Let $D = \{1, 2, \ldots, 9, 10\}$. Let $(\Omega, \mathbb{P})$ be a probability space, and let $R, S : \Omega \to D$ be random variables.

We say that R and S are *independent* random variables if for all $r, s \in D$, the events $R = r$ and $S = s$ are independent.

We say that R is *uniform* if for every $d \in D$, $\mathbb{P}[R = d] = \frac{1}{|D|}$.

Finally, we define the event $R = S$ to be the set of outcomes in Ω on which R and S take equal values:

$$R = S \;=\; \{\omega \in \Omega \mid R(\omega) = S(\omega)\}\,.$$

Suppose that R and S are independent and R is uniform. We claim that $\mathbb{P}[R = S] = \frac{1}{|D|}$. Intuitively, it seems whatever value S happens to take, R is just as likely to take that value as any other. But this is not a rigorous argument! Write a careful proof of this claim.

HINT: Can you write the event $R = S$ as a union of disjoint events?

Solution:

We can write
$$R = S \;=\; \bigcup_{d \in D} \{\omega \in \Omega \mid R(\omega) = d \wedge S(\omega) = d\}\,.$$
This is a union of disjoint events, since R and S take on a distinct value in each event.

We compute

$$\begin{aligned}
\mathbb{P}[R = S] &= \mathbb{P}[\bigcup_{d \in D} \{\omega \in \Omega \mid R(\omega) = d \wedge S(\omega) = d\}] \\
&= \sum_{d \in D} \mathbb{P}[\{\omega \in \Omega \mid R(\omega) = d \wedge S(\omega) = d\}] \\
&= \sum_{d \in D} \mathbb{P}[R = d] \cdot \mathbb{P}[S = d] \text{ (since } R \text{ and } S \text{ are independent)} \\
&= \sum_{d \in D} \frac{1}{|D|} \mathbb{P}[S = d] \text{ (since } R \text{ is uniform)} \\
&= \frac{1}{|D|} \sum_{d \in D} \mathbb{P}[S = d]
\end{aligned}$$

$$= \frac{1}{|D|}$$

Problem 4 (Mind Bender — *Extra Credit*)

This week's Mind Bender is a Lean question!

The problem can be found by navigating to `BrownCs22/Homework/Homework09.lean` in the directory browser on the left of your screen in your Codespace. The comment at the top of that file provides more detailed instructions.

You will submit your solution to this problem separately from the rest of the assignment. Once you have solved the problem, download the file to your computer (right-click on the file in the Codespace directory browser and click "Download"), and upload it to Gradescope.