Homework 2
Due: Friday, February 18

All homeworks are due at 11:59 PM on Gradescope.

Please do not include any identifying information about yourself in the handin, including your Banner ID.

Be sure to fully explain your reasoning and show all work for full credit.

\LaTeX tips

As you begin using \LaTeX, here are some useful commands that might come in handy for this homework:

\begin{itemize}
  \item \texttt{\textbackslash{Pow}} \texttt{\mathcal{P}} \texttt{\textbackslash{cap}} \texttt{\cap} \texttt{\textbackslash{cup}} \texttt{\cup}
  \item \texttt{\textbackslash{star}} \texttt{\ast} \texttt{\overline{\{A\}}} \texttt{\overline{A}} \texttt{\emptyset}
  \item \texttt{\textbackslash{times}} \texttt{\times} \texttt{\setminus} \texttt\textbackslash{\subset}
\end{itemize}

A note that \LaTeX is not required until Homework 3.

Problem 1

a. Prove (using the “set element” method) or disprove the following claim:
   For arbitrary sets $A$ and $B$, $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.

b. Prove (using the “set element” method) or disprove the following claim:
   For arbitrary sets $A$ and $B$, $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$. 

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Problem 2

a. A binary operation, $\star$, is an operation on a set that takes in two elements from the set and returns a third.
We say that a binary operation $\star$ on a set $S$ is commutative if
\[ x \star y = y \star x \text{ for all } x, y \in S. \]

Example
For example, the addition operation (+) and the subtraction operation (−) are binary operations on the integers. Addition is commutative while subtraction is not.

Let $X$ be a finite set. For each of the following operations, prove whether or not the given operation is commutative over $\mathcal{P}(X)$.

i. Set union
ii. Set intersection
iii. Set difference
iv. Symmetric difference\(^1\)

b. Consider a binary operation $\star$ on a set $S$. An identity element for $\star$ is any $e \in S$ such that $e \star x = x \star e = x$ for all $x \in S$.

Example
For example, 0 is an identity element for the operation + over the integers, because $x + 0 = 0 + x = x$ for all $x \in \mathbb{Z}$.

Let $X$ be a finite set. Which elements in $\mathcal{P}(X)$ are identity elements for the set union operation ($\cup$)? Which elements in $\mathcal{P}(X)$ are identity elements for the set intersection operation ($\cap$)? Prove your response. That is, if identity elements exists, prove that the identity elements you found are indeed identity elements, and prove that they are the only identity elements (i.e. that no other element can be an identity). If they do not exist, prove that no such identity exists.

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\(^1\)Recall that the symmetric difference is defined as $(A \cup B) \setminus (A \cap B)$. See the lecture notes and/or text for definitions of any other operations.
Problem 3

a. Given the following sets:

\[ A = \{a, b, c, d, g, f\} \]
\[ B = \{a, e\} \]
\[ C = \{b, d, e, g\} \]
\[ D = \{a, g\} \]

i. Compute \((A \cap C) \cup D\)

ii. Compute \(B \times ((A \cap C) \cup D)\)

iii. Compute \(|\mathcal{P}(B \times ((A \cap C) \cup D))|\)

b. Let \(A, B, C\) be arbitrary sets, and assume they are all subsets of a universal set \(U\). Using set algebra, show that the following equivalences hold:

i. \((A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)\)

ii. \(\overline{A \cup (B \cap C)} = (\overline{C} \cup B) \cap A\)

iii. \((A \cap B) \cup (B \cap C) \cup (C \cap A) = (B \cap (A \cup C)) \cup (C \cap (A \cup B))\)

**Hint:** Try gaining insight by going in the reverse direction.
Problem 4 (Mind Bender — Extra Credit)

Recall that we can create sets with descriptions using set-builder notation, like

\[ A = \{ x \in \mathbb{R} \mid x^2 > 1 \}. \]

We would read this as “\( A \) is the set of \( x \) in the real numbers such that \( x^2 > 1 \).” The condition that \( x^2 > 1 \) is a condition we place on this set to define it. Another example could be

\[ H = \{ h \mid h \text{ is a hot drink} \}. \]

to which \( H = \{ \text{hot coffee, hot tea, latte, \ldots } \} \). This is an example of a description that is in natural language. One might consider if every description is a valid description.

Consider the description “\( X \) does not contain itself”.

Example

For example, let

\[ H = \{ h \mid h \text{ is a hot drink} \}. \]

be the set of all hot drinks. This set \( H \) does not contain itself (\( H \not\in H \)) since the set of all hot drinks is not a hot drink.

However, let

\[ J = \{ h \mid h \text{ is not a hot drink} \}. \]

be the set of everything that isn’t a hot drink. Well, \( J \) is a set of things, that clearly isn’t a hot drink, so \( J \in J \) and we say \( J \) “contains itself”.

Let’s build a set with this specific description. Let \( S \) be the set that contains all sets that do not “contain themselves”. That is, we define it to be

\[ S = \{ X \text{ is a set} \mid X \not\in X \}. \]

A set such as \( J \) would not be in \( S \), as \( J \) contains itself. However, a set such as \( H \) would be in \( S \), as \( H \) does not contain itself.

a. Show that the assumption that \( S \) is a member of \( S \) leads to a contradiction.

b. Show that the assumption that \( S \) is not a member of \( S \) leads to a contradiction.

c. What do these contradictions suggest about how we can or cannot define a set? This paradox is called Russell’s Paradox. Are there any ways to resolve these contradictions in set theory? Do some research and cite at least one source.\(^2\)

\(^2\)This is an open ended question, and there is no right answer! You should demonstrate to us that you have an understanding of what is going on.