Problem 1

Assume there are 101 hot drinks, each of which has some nonnegative, integer number of marshmallows. Prove that it is possible to choose 11 of them whose total number of marshmallows is divisible by 11.

**Hint:** Associate each hot drink to its number of marshmallows modulo 11, and consider two cases: one where there is a hot drink in each of the categories and one where there isn’t.
Problem 2

Consider the following equation:

\[ x_1 + x_2 + x_3 + x_4 = 75 \]

where each \( x_i \) must be a non-negative integer.

a. Count the number of solutions to this equation.

b. Now suppose we require a solution with \( x_1 \) and \( x_3 \) strictly positive. Count the number of solutions under this new constraint.

c. More generally, suppose we require a solution where some \( a_i \) are fixed constant nonnegative integers and \( x_i \geq a_i \) (that is, we have a constant \( a_i \) corresponding to each \( x_i \)), satisfying

\[ \sum_{i=1}^{4} x_i \leq 75. \]

That is, each \( x_i \) is restricted to be at least \( a_i \). Again, count the number of solutions under this new constraint. You can give this as an expression in terms of the \( a_i \)'s. You may use summation (\( \Sigma \)) notation in your final answer to express this count.

Problem 3

Prove the following equality.

Do not use any algebraic manipulation in your argument. Instead, give a “counting argument”: why is the number of ways to chose \( k \) objects from \( n \) options the same as the sum shown on the right?

\[ \binom{n}{k} = \binom{n-2}{k} + 2 \binom{n-2}{k-1} + \binom{n-2}{k-2} \]
# Problem 4 (Mind Bender — *Extra Credit*)

*Mind Benders* are extra credit problems intended to be more challenging than usual homework problems and are an exploration into a topic not covered in lecture. This week, we have an exploration into *generating functions*.

We define a sequence of numbers as a list of (indexed) numbers:

$$a_0, a_1, a_2, a_3, a_4, \ldots$$

A sequence can be finite\(^1\) or infinite, can have repeats, and need not be in increasing or decreasing order.

## Example

Here are some sequences:

i. The constant 1 sequence: 1, 1, 1, 1, \ldots

ii. The sequence of positive integers: 1, 2, 3, 4, 5, \ldots

iii. The Fibonacci numbers: 1, 1, 2, 3, 5, 8, \ldots

iv. \(\binom{6}{i}\) where \(a_i = \binom{6}{i}\):

<table>
<thead>
<tr>
<th>(\binom{6}{0})</th>
<th>(\binom{6}{1})</th>
<th>(\binom{6}{2})</th>
<th>(\binom{6}{3})</th>
<th>(\binom{6}{4})</th>
<th>(\binom{6}{5})</th>
<th>(\binom{6}{6})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,</td>
<td>6,</td>
<td>15,</td>
<td>20,</td>
<td>15,</td>
<td>6,</td>
<td>1,</td>
</tr>
</tbody>
</table>

*Generating functions* are functions (in \(x\)) that generate a sequence as the coefficients of the powers of \(x\).

## Example

For example,

$$f(x) = 3 + 2x + 3x^2 + x^4$$

$$= 3x^0 + 2x^1 + 3x^2 + 0x^3 + 1x^4 + 0x^5 + 0x^6 + \cdots$$

is the generating function for the sequence 3, 2, 3, 0, 1, 0, 0, \ldots.

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\(^1\)If a sequence is finite, we assume there are infinitely many zeros at the end.
In general,

\[ f(x) = \sum_{i=0}^{n} a_i x^i \]

is the generating function for sequence \(a_0, a_1, a_2, \ldots\). 

a. Using the binomial theorem, verify that the function

\[ b(x) = (1 + x)^6 \]

generates sequence (iv) from the example above. That is, it generates the sequence \(a_0, a_1, a_2, \ldots\) where \(a_i = \binom{6}{i}\).

Can you generalize this if you replace 6 with an arbitrary \(n\)?

b. Let’s say our sequence \(a_i\) is the number of ways to make \(i\) cents with 2 one-cent coins, 3 nickels, 2 dimes, and 1 quarter. It starts as

\[
\begin{align*}
a_0, & \quad a_1, \quad a_2, \quad a_3, \quad a_4, \quad a_5, \quad a_6, \ldots \\
1, & \quad 1, \quad 1, \quad 0, \quad 0, \quad 1, \quad 1, \ldots 
\end{align*}
\]

Justify that

\[ f(x) = (1 + x + x^2)(1 + x^5 + x^{10} + x^{15})(1 + x^{10} + x^{20})(1 + x^{25}) \]

is the generating function for this sequence.

How many ways can we make 26 cents? Use a tool like WolframAlpha to expand the polynomial and check that the coefficient of \(x^{26}\) in the polynomial is what you got.

c. The generating function for sequence (i) from above is

\[ f(x) = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \]

Consider the generating function

\[ g(x) = \frac{1}{(1-x)^4} = (1 + x + x^2 + x^3 + \cdots)^4 \]

Use a computer to (Taylor-series) expand this function as a polynomial. Explain why the coefficient of \(x^{75}\) gives the solution to 2a from this homework.

Then, count the number of solutions to \(x_1 + x_2 + x_3 = 50\) (where each \(x_i\) is a nonnegative integer) using generating functions.

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2Check that this makes sense before beginning.

3There is only one way to make 0 cents (no coins at all). There is 1 way each to make 1 and 2 cents (we use the one-cent coins); we can’t make 3 or 4 cents with our coins; we can make 5 cents with a nickel; we can make 6 cents with nickel plus one-cent.

4This comes from the Taylor series expansion of \(f\) about 0.