Homework 9

Due: Friday, April 22

All homeworks are due at 11:59 PM on Gradescope.

Please do not include any identifying information about yourself in the handin, including your Banner ID.

Be sure to fully explain your reasoning and show all work for full credit.

Problem 1

a. You have 4 drinks:
   i. Drink 1 is iced, not lemon-flavored, and not carbonated.
   ii. Drink 2 is not iced, is lemon-flavored, and not carbonated.
   iii. Drink 3 is not iced, not lemon-flavored, but is carbonated.
   iv. Drink 4 is iced, lemon-flavored, and carbonated.

You pick one drink from these four, with uniform probability. Define the event $X_1$ as picking an iced drink, $X_2$ as picking a lemon-flavored drink, and $X_3$ as picking a carbonated drink.

Prove that $X_1$, $X_2$, and $X_3$ are each pairwise independent, but not mutually independent.

b. (The following subpart is independent\(^1\) of part a.)

Consider the following random variables:

$X$, which takes values \{1, 2, 3, 4, 5\} with uniform probability.

$Y$, which takes values \{0, 1, 2\} with the following distribution:

\[
\Pr(Y = 0) = \frac{1}{4} \quad \Pr(Y = 1) = \frac{1}{4} \quad \Pr(Y = 2) = \frac{1}{2}
\]

Calculate the following:

i. $\mathbb{E}(X)$

ii. $\mathbb{E}(Y)$

iii. $\mathbb{E}(X + Y)$

iv. $\mathbb{E}(X \cdot Y)$

\(^1\)Get it?
Problem 2

Buck, Joe, and Pete are all getting hot chocolate at their favorite café. To spice up their lives, they decide to ask the barista to add to each of their hot chocolates some number of marshmallows uniformly distributed between 1 and 5.

a. What is the probability that the maximum number of marshmallows someone added to their hot chocolate is 1? What about 2?

b. What is the expected value of the maximum number of marshmallows added to one of the three cups of hot chocolate?

Hint: You’ve already done some of the work for this by calculating the answer to part a. Also, think about how you can use the number of ways to get a maximum of \( n \) marshmallows to calculate the number of ways to get a maximum of \( n + 1 \) marshmallows.

c. What is the expected value of the minimum number of marshmallows added to one of the three cups of hot chocolate?

Hint: Reuse all the probabilities you calculated in part b.

d. What is the expected value of the difference between the maximum and minimum number of marshmallows added to one of the three cups of hot chocolate?

Problem 3

The 4 HTAs bring 6 lucky UTAs to get hot beverages. The 10 TAs each order a different drink for themselves. However, the clumsy barista mixes up their orders such that all the drinks got shuffled!

What is the probability that at least one of the 4 HTAs gets the drink that they ordered? (Assume that all shufflings are equally likely.)
Problem 4 (Mind Bender — Extra Credit)

This problem asks you to think a bit about the psychology and philosophy of probability—and how that interacts with math!

We mentioned briefly in lecture two “interpretations” of probability, the frequentist and Bayesian interpretations. To a frequentist, probability statements only apply to repeatable events. Saying that the probability a weighted coin comes up heads is .7 means that, if you flip it lots of times, about 70% of the flips will be heads. To a Bayesian, probability statements measure subjective degrees of belief: the same statement means that the speaker is 70% certain a flip will be heads.

While not inherently about gambling, it’s common to describe the Bayesian interpretation in terms of bets: a Bayesian should be willing to bet, at 7:3 odds for, that the coin will be heads. (That means the Bayesian will win $3 if the coin is heads, and lose $7 if it is tails.)

This is all nice in theory. In practice, humans are very bad at computing complex probabilities.

a. Read the article on “long odd” events here². Note the language it uses, and how similar it is to the language we’ve used in class! The article concludes, from the popularity of bets with low expected payoffs, that “the average soccer fan lacks the skills to accurately judge complex gambles.”

Do you agree with this analysis? Are you aware of other real-life scenarios where people have similar trouble? Could there be alternate explanations for why people are willing to make these bets?

We’ve touched on the notion of coherence a little bit in class. If $A$ and $B$ are independent events, then $\Pr(A \land B)$ must be $\Pr(A) \cdot \Pr(B)$. But as the article above gets at, and noted elsewhere, people tend to overestimate the likelihood of the conjunction of two rare events, even when they correctly know the likelihood of each event individually. Their Bayesian model doesn’t follow the laws of probability we’ve laid out!

When interpreted as “willingness to bet,” this incoherence can be exploited. Not just by offering overpriced bets, as in the article above—in some scenarios incoherent bettors can be vulnerable to guaranteed losses.

²https://behavioralscientist.org/gambling-dark-side-nudges/
b. I’m going to flip two coins. Let $A$ be the proposition that the first coin lands on heads, and $B$ be the proposition that the second coin lands on heads.

I’m willing to give you 2:1 odds against any logical combination of these propositions. For example:

- You could stake $10 on the proposition $A$. If the first coin lands on heads, I pay you $20; tails, you pay me $10.
- You could stake $50 on the proposition $A \land B$. If both coins land on heads, I pay you $100; otherwise, you pay me $50.
- You could stake $10 on $A$ and $20 on $\neg A \land B$. If the first coin comes up heads, you’ll break even, since the first bet will pay $20 and the second will cost $20. But if both coins come up tails, you lose $30. If the first is tails and the second is heads, you lose the first bet but win the second, for a gain of $30. Notice you can’t win both bets at once!

Can you find a proposition—or a set of propositions—to bet on that will give you a guaranteed profit?

If you’re interested and would like to know more, these types of gambles are sometimes known as Dutch books\(^3\).

\(^3\)[https://en.wikipedia.org/wiki/Coherence_(philosophical_gambling_strategy)]