Functions, Injectivity, Surjectivity, Bijections

Robert Y. Lewis

CS 0220 2022

February 16, 2022
Overview

1. Relation Diagrams (4.4.1)

2. Relational Images (4.4.2)
Binary relations

Definition. A binary relation, $R$, consists of a set, $A$, called the **domain** of $R$, a set, $B$, called the **codomain** of $R$, and a subset of $A \times B$ called the **graph** of $R$.
A binary relation:

- is a *partial function* when it has the \([\leq 1 \text{ arrow out}]\) property.  
  Book: “function”. Us: “function” is \([= 1 \text{ arrow out}]\) property.
- is *surjective* when it has the \([\geq 1 \text{ arrows in}]\) property.
- is *total* when it has the \([\geq 1 \text{ arrows out}]\) property.
- is *injective* when it has the \([\leq 1 \text{ arrow in}]\) property.
- is *bijective* when it has both the \([= 1 \text{ arrow out}]\) and the \([= 1 \text{ arrow in}]\) properties.
Example relation #1

partial function: \(\leq 1\) out. surjective: \(\geq 1\) in. total: \(\geq 1\) out. injective: \(\leq 1\) in.
bijective: \(= 1\) out and \(= 1\) in.

Partial function; surjective; total. Not injective, not bijective.
Summary: a surjective function. (Implies partial function and total.)
Example relation #2

partial function: \( \leq 1 \text{ out} \). surjective: \( \geq 1 \text{ in} \). total: \( \geq 1 \text{ out} \). injective: \( \leq 1 \text{ in} \).

bijective: \( = 1 \text{ out} \) and \( = 1 \text{ in} \).

Partial function; total; injective. Not surjective, not bijective.
Summary: an injective function. (Implies partial function and total.)
Example relation #3

partial function: $\leq 1$ out. surjective: $\geq 1$ in. total: $\geq 1$ out. injective: $\leq 1$ in.
bijective: $= 1$ out and $= 1$ in.

Equation $y = 1/x^2$ on $\mathbb{R}^+$. $x$ is an element in the domain, $y$ is an element in the co-domain.

Partial function; surjective; total; injective; bijective.
Summary: a bijective (partial) function. (Implies everything else.)
Example relation #4

partial function: \([\leq 1 \text{ out}].\) surjective: \([\geq 1 \text{ in}].\) total: \([\geq 1 \text{ out}].\) injective: \([\leq 1 \text{ in}].\)

bijective: \([= 1 \text{ out}]\) and \([= 1 \text{ in}].\)

Equation \(y = 1/x^2\) on \(\mathbb{R}\).

Partial function. Not anything else.
**Image definition**

Definition. The *image* of a set $Y \subseteq A$ under a relation $R : A \rightarrow B$, written $R(Y)$, is the subset of elements of the codomain $B$ of $R$ that are related to some element in $Y$.

In terms of the relation diagram, $R(Y)$ is the set of points with an arrow coming in that starts from some point in $Y$.

$$R(Y) = \{x \in B \mid \exists y \in Y, y R x\}.$$
Inverse definition

Definition: The inverse $R^{-1}$ of a relation $R : A \rightarrow B$ is the relation from $B$ to $A$ defined by the rule

$$b R^{-1} a \iff a R b.$$ 

Definition: The image of a set under the relation $R^{-1}$ is called the inverse image of the set. That is, the inverse image of a set $X$ under the relation $R$ is defined to be $R^{-1}(X)$.

Example: $x R y$ iff there’s a dictionary word with first letter $x$ and second letter $y$. The image $R(\{c, k\})$ is the letters that can appear after ‘c’ or ‘k’ at the beginning of a word. It’s the set $\{a, e, h, i, l, n, o, r, u, v, w, y, z\}$.

The inverse image $R^{-1}(\{c, k\})$ is the letters that can appear before ‘c’ or ‘k’ at the beginning of a word. It’s the set $\{a, e, i, o, s, t, u\}$.
Inverses of relations

What can we infer about $R^{-1}$ if $R$ is:

- partial function? injective
- surjective? total
- total? surjective
- injective? partial function
- bijective? bijective
- function? injective and surjective
More examples to consider

Make natural examples for each combination of properties.

- $y = \sqrt{x}$ on $\mathbb{R}$
- $y = \sqrt{16 - \sqrt{x}}$ on $\mathbb{R}$
- $y = |x + 10|$ on $\mathbb{Z}$
- $y = |x \mod 2|$ on $\mathbb{Z}$
- $y = \sin(x)$ on $\mathbb{R}$
- $x^2 + y^2 = 10$ on $\mathbb{R}$