Proofs About Sets & Quantification

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Overview

1. Proofs about Sets, Proof by Cases (1.7)

2. Predicate Formulas (3.6)
Proving set equalities: set-element method

Theorem: For any sets $A$ and $B$ of elements in universe $U$, $A \cap B = \overline{A} \cup \overline{B}$.

DeMorgan’s Law again! Now, connects intersection and union instead of $\land$ and $\lor$.

An object $x \in A \cap B$ if it is not in both $A$ and $B$. Such an element must either not be in $A$ or not be in $B$. It follows that such an element must be in $\overline{A} \cup \overline{B}$. Thus, we have $A \cap B \subseteq \overline{A} \cup \overline{B}$.

Note also that an object $x \in \overline{A} \cup \overline{B}$ if it is either not in $A$ or it is not in $B$. Such an element can’t be in both $A$ and $B$, therefore. Said another way, it must be in $A \cap B$. Thus, we have $\overline{A} \cup \overline{B} \subseteq A \cap B$.

Since both $A \cap B \subseteq \overline{A} \cup \overline{B}$ and $\overline{A} \cup \overline{B} \subseteq A \cap B$ are true, we know $A \cap B = \overline{A} \cup \overline{B}$. 
Fact about groups of people

Any two people have either met or not.

Given a group of people $G$, if all pairs of people in $G$ have met, we’ll call it a club. If no two people in $G$ have met, we’ll call them strangers.

**Theorem.** Every collection of 6 people includes a club of 3 people or a group of 3 strangers.

Does that seem true? Try some examples on the board.
Proof (Part 1)

The proof is by case analysis. Let $x$ denote one of the six people. Let $R = G - \{x\}$ be the rest. There are two cases:

1. Among $R$, at least 3 have met $x$.
2. Among $R$, at least 3 have not met $x$.

At least one of these cases must hold. Since $|R|$ is odd, either more than half in $R$ know $x$ or less than half in $R$ know $x$ (and therefore more than half do not know $x$).

Case 1: At least 3 have met $x$. Let $J \subseteq R$ be those individuals. Two subcases:

1.1 No pair in $J$ have met each other. So, $J$ is a group of at least 3 strangers and the theorem holds in this subcase.
1.2 Some pair in $J$ have met each other. That pair and $x$ are a club of 3 people and the theorem holds in this subcase, too.

That covers Case 1!
Proof (Part 2)

Case 2: At least 3 have not met \(x\). Let \(J \subseteq R\) be those individuals. Two subcases:

2.1 Every pair in \(J\) have met each other. So, \(R\) is a club of at least size 3 and the theorem holds in this subcase.

2.2 Some pair in \(J\) haven’t met each other. That pair and \(x\) are a group of strangers of 3 people and the theorem holds in this subcase, too.

That covers Case 2! It’s kind of the inverse-video version of Case 1.

Since we showed that only these two cases can occur and the theorem holds in both, the theorem *always* holds.
Quantifiers, Revisited

**Always True** (universal quantification)

\[ \forall x \in \mathbb{R}, x^2 + 1 \geq 0. \]
- For all \( x \in D \), \( P(x) \) is true.
- \( P(x) \) is true for every \( x \) in the set \( D \).

**Sometimes True** (existential quantification)

\[ \exists x \in \mathbb{Z}, x \text{ is even and } x \text{ is prime}. \]
- There is an \( x \in D \) such that \( P(x) \) is true.
- \( P(x) \) is true for some \( x \) in the set \( D \).
- \( P(x) \) is true for at least one \( x \in D \).
Mixing quantifiers

**Theorem** (sparse squares): There’s a perfect square arbitrarily far from its closest perfect square.

Clear? Maybe a tad vague. True? How say in math?

\[ \forall d \in \mathbb{N}, \exists i \in \mathbb{N}, \forall j \in \mathbb{N}, \ (i \text{ is a perfect square}) \land (|i - j| \leq d \rightarrow \neg(j \text{ is a perfect square})). \]

The expressions nest inside each other. The order matters.

You can think of it like a little game. I’m claiming that you can pick any \( d \) you want. I’ll then pick an \( i \) that’s a perfect square AND no matter what \( j \) you pick that is within \( d \) values of \( i \), \( j \) won’t be a perfect square.

So, what’s my winning strategy?
Any ambiguity is too many

“If you can identify any bird, you’ve got a talent.”
1. If $\exists b$, you can identify $b$, then you’ve got a talent.
2. If $\forall b$, you can identify $b$, then you’ve got a talent.

“...statistics show that, in the UK, some brews a cup of tea every second. That person’s name is Nigel.”
1. $\forall t, \exists p$, $p$ brews a cup of tea at second $t$
2. $\exists p, \forall t$, $p$ brews a cup of tea at second $t$
More examples

“The whole article is not available.”

- not $\forall$ article part $x$, $x$ is available
- $\forall$ article part $x$, $x$ is not available

Data privacy: consider an “anonymous” survey with a number of questions. How to pool the responses?

- For each response $r$, for each answer $a$ on $r$, there is a person $p$ who gave answer $a$.
- For each response $r$, there is a person $p$ such that for each answer $a$ on $r$, $p$ gave answer $a$. 
DeMorgan returns: Negating quantifiers

These two statements are equivalent:

- Not everyone likes coffee.
- There’s someone who doesn’t like coffee.

\( \neg \forall x, P(x) \) is equivalent to \( \exists x, \neg P(x) \).
Assertion about predicates

The formula $\exists x, \forall y, P(x, y)$ implies the formula $\forall y, \exists x, P(x, y)$.

If $\exists x, \forall y, P(x, y)$, there must be some specific $x^*$ such that $\forall y, P(x^*, y)$. As a result, $\forall y, \exists x, P(x, y)$ because we can always choose $x^*$ to be the selected $x$.

On the other hand, $\forall y, \exists x, P(x, y)$ does not imply $\exists x, \forall y, P(x, y)$.

Let’s come up with a counterexample: a property $P$ where the first formula is true and the second is false.

We can describe $P$ by a table saying for which values of $x$ and $y$ $P(x, y)$ is true:

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