Introduction to Graph Theory

A graph is two related sets: $V(G)$, the vertex set, and $E(G)$, the edge set. Each element of $E(G)$ is a set containing exactly two elements of $V(G)$.

We often visualize graphs by drawing the vertices as dots and the edges as lines between them. Here is an example of a graph with vertex set $\{A, B, C, D, E, F\}$ and edge set $\{\{A, B\}, \{B, C\}, \{A, E\}, \{C, E\}, \{A, D\}, \{C, F\}, \{D, E\}, \{E, F\}\}$.

Note that we have defined an edge as a set of cardinality two: this means that a vertex cannot have an edge to itself, and there is no sense of direction in edges. Additionally, since edges are contained in a set, there can be at most one edge between any two vertices.

- If $u, v \in V(G)$, $u$ is adjacent to $v$ if $\{u, v\} \in E(G)$. (Note that by this definition a vertex is not adjacent to itself).

- The degree of a vertex is a count of the number of vertices it is adjacent to. Formally, for a vertex $v$, $\deg(v) = |\{u|\{v, u\} \in E(G)\}|$.

- The empty graph on $n$ vertices has an empty edge set.

- The complete graph on $n$ vertices $K_n$ has all possible edges between vertices, that is, $E(G) = \{(u, v)|u, v \in E(G), u \neq v\}$. Every pair of vertices is adjacent.
• A subgraph $G'$ of a graph $G$ is a graph such that $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. Since $G'$ is a graph, each edge in $E(G')$ must be between two vertices in $V(G')$.

**Task 1**

Let $G$ be a graph such that $V(G) = \{a, b, c, d, e, f\}$ and $E(G) = \{\{b, c\}, \{a, b\}, \{d, e\}, \{c, d\}, \{d, b\}\}$.

1. Draw $G$.

2. What is the degree of each vertex in $G$?

3. What are the possible simple paths from $a$ to $e$?

4. Say we take a vertex $v$ uniformly at random from $V(G)$. Let $X$ be a random variable that represents the degree of $v$. What is the probability distribution of $X$?

5. What is $\mathbb{E}[X]$?

6. Optional: Can you develop a general expression for $\mathbb{E}[X]$ in terms of $|V|$ and $|E|$?
Definitions

- A **path** is a sequence of vertices, consecutive elements of which are endpoints of edges in the graph.

- A **simple path** is a path in which no vertex occurs more than once.

- A **cycle** is a path in which the first and last vertex are the same

- A **simple cycle** is a cycle in which all but the last vertex form a simple path.

- A **Hamiltonian tour** is a simple cycle that includes all the vertices in the graph.

- Two vertices are **connected** if there exists a path between them (or if they are the same vertex).

- An **Eulerian tour** is a cycle that visits every edge exactly once.

- A **Bipartite graph** (optional) is a graph whose vertex set can be partitioned to two parts, and no edge connects a pair of vertices belonging to the same part. Any graph that is a subgraph of a complete bipartite graph is bipartite.

- Let $G = (V, E)$ and $G' = (V', E')$. $G$ and $G'$ are **isomorphic** (optional) if there is a bijection $f : V \rightarrow V'$ such that for every pair of vertices $\{x, y\} \in E$ if and only if $\{f(x), f(y)\} \in E'$. In other words, two graphs are isomorphic if we can find a consistent relabeling of the edges and vertices of the two graphs.
Task 2

Consider the following map of planets (labelled A–G) and the routes between them:

a. A coffee addict wants to explore all the routes between a number of cafes. Can a cycle be found which traverses each route only once? Particularly, find a path that starts at A, goes along each route exactly once, and ends back at A.

In other words, find an Euler tour for the graph!

b. A coffee addict wants to visit a number of cafes. Can a cycle be found that visits each cafe exactly once? Particularly, find a path that starts at A, goes to each cafe exactly once, and ends back at A.

In other words, find a Hamiltonian tour for the graph!
Task 3

a. Given a certain labelling of the vertices, how many distinct graphs can be made with 4 vertices?

b. For any graph $G$, explain why

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|.$$ 

c. Tim Horton’s is having an event where two customers that purchase a pastry to split receive 10% off their order. 35 CS22 students are planning on taking advantage of this offer by having each student split pastries with 3 other students. Prove that it is impossible to do so.

d. Optional: Prove that in any graph there exist two vertices with the same degree. 

Hint: What are the possible degrees of a vertex in a graph with $n$ nodes?

Checkpoint 1 — Call over a TA!
Trees!

- A **cycle** is a path that starts and ends with the same vertex.
- A graph is **cyclic** if it contains a simple cycle and **acyclic** otherwise.
- A graph is **connected** if there is a simple path between each pair of vertices (that is, all vertices are connected).
- A **tree** is a connected, acyclic graph.
- A **forest** is an acyclic graph. In other words, a forest is a set of trees.
- A **leaf** of a tree is a vertex with degree 1.
- A **spanning tree** of a graph $G$ is a subgraph $T$ of $G$ such that $V(T) = V(G)$ and $T$ is a tree.

**Task 4**

In 1997, a chess-playing computer program called Deep Blue beat the reigning world champion of chess, Garry Kasparov, in a 6-round series. Although it was unclear whether computers can reason as well as humans, the victory did establish that computers are better at evaluating complex game trees, the basis of computer-based game strategies.

A game tree shows all possible playing strategies of both players in a game. Each vertex in the tree represents a configuration of the game as well as an indication of whose turn it is to play next. We illustrate game trees in a much simpler context of tic-tac-toe.

The first three layers of the game tree of tic-tac-toe

The **root** of the tree is the initial configuration of the game, which is an empty grid in the case of tic-tac-toe. The **children** of a configuration $c$ are all the configurations that can be reached from $c$ by a single move of the correct player. A configuration
is a **leaf** vertex in the tree if the game corresponding to that configuration is over (won, lost, tied).

Here, we see a subtree near the end of the game:

1. How many leaf nodes are in the subtree drawn above?

2. What is the length of the longest path from a root to a leaf in a tree represented by the full tic-tac-toe game?

3. **Optional:** What is the length of the longest path between any two vertices in a tree represented by the full tic-tac-toe game?
Task 5

1. Prove that, in a tree, there is exactly one simple path between any two vertices.

2. One way of defining a tree is as a **minimally connected, maximally acyclic graph**. That is because if we remove an edge, our graph will no longer be connected and if we add an edge, the graph will no longer be acyclic. Prove these two statements.

3. Optional: Let $T$ be a tree with at least two vertices. Prove that $T$ has at least two leaves. **Hint:** Start by assuming that the longest simple path in $T$ does not start and end with leaves. What contradiction can we reach?
Induction on Graphs

Many graph theory questions involve induction. When doing induction on graphs, we often use a technique referred to in CS0220 as “build-down” induction. It is just regular induction, with a little extra care taken to make sure it applies to objects more complicated than integers. To review, here are the steps of an inductive proof:

1. Define the predicate \( P(n) \).
2. Show that the base case is true.
3. Assume the inductive hypothesis is true. If you are using standard induction, then you will assume \( P(k) \) is true for some integer \( k \). If you are using strong induction, then you will assume \( P(i) \) is true for all \( i \leq k \).
4. Show that \( P(k+1) \) is true given the inductive hypothesis.

   Here is where build-down induction differs from our usual perspective. A lot of the time, we start by invoking our inductive hypothesis, and manipulating the \( k \) case to get the \( k+1 \) case. In build-down induction, we do the following instead:

   i. Consider an arbitrary \( k+1 \) case.
   ii. Alter the \( k+1 \) case to obtain a \( k \) case.
   iii. Invoke the induction hypothesis, asserting that your property holds for the \( k \) case.
   iv. Recover the original \( k+1 \) case by undoing your alteration.
   v. Prove that the property still holds despite building back up to the original \( k+1 \) case.

This technique is especially useful when it is not clear how to address all \( k+1 \) cases in general when starting from a \( k \) case. For example, if we started with a graph with \( k \) edges, where do we add an edge to obtain a general graph with \( k+1 \) edges? There are a lot of ways to add this extra edge that we have to consider. To avoid this ambiguity, we can start with an arbitrary \( k+1 \) case. Then, we build down to a \( k \) case, at which point we can invoke our induction hypothesis. Then, we build back up to the original \( k+1 \) case.

Try it out on a few sample problems!
Task 6

1. Prove that a tree with $n$ vertices has $n - 1$ edges.

2. Optional: Prove that $K_n$ has $\frac{n(n-1)}{2}$ edges. Note that we could do this without build-down induction, but you should use it here for practice.
Graph Coloring

Maps are often drawn such that no two adjacent regions are the same color. This makes it easy when looking at the map to see the distinct regions. For instance, here is a map of the United States where no adjacent states are the same color:

![United States Map]

You might notice that this map was drawn with only 4 distinct colors. There are 50 U.S. states, so it might be a bit surprising that it is possible to use only four colors. This observation leads to a question that cartographers and mathematicians have asked for centuries: given a map, how many colors will we need to draw it?

Like so much else in life, this problem is really graph theory in disguise! Any map can be easily converted to a graph.

For instance, we can represent this map of a region of Europe with the displayed graph. The graph was created by making each country into a vertex, and drawing an edge between two vertices if the countries share a border (and therefore need distinct colors in the map).

![Europe Map and Graph]

Task 7

1. Consider the map of states and territories of Australia, which can be represented as the following graph:
Determine the minimum number of colors we need to properly color this graph, and show one such possible coloring.

2. Optional: Now, consider the map of countries in South America, which can be represented as the following graph:

Determine the minimum number of colors we need to properly color this graph, and show one such possible coloring.
Optional: Isomorphism

Task 8

Two graphs are isomorphic if there exists a matching or relabelling of the vertices of the graphs.

The two graphs above are isomorphic because we can match a with 0, b with 3, c with 1, d with 4, and e with 2. Given this matching of the vertices, it’s easy to check that the vertices are connected the same way in the graphs. Yet, it’s hard to determine if two graphs are actually isomorphic! Let’s try this ourselves:

1. Out of the three graphs below, determine which two are isomorphic.
   
   Hint: Which of these are bipartite graphs?
2. Let’s try an even harder example! Determine which two out of these four graphs are isomorphic.

Checkoff — Call over a TA!